MATC44 Week 4 Notes

Principle of Invariants:

a) Theory:

- The principle of invariants is applicable to problems which involve a process and the goal is to see what possible final states of that process are.

- A general statement of the principle is the following:

  1. Assume we have a process which takes place in discrete steps $S_1, S_2, \ldots, S_k$ where S_k is the final step of the process.

  2. Assume that the vars of the problem take different values at diff steps.

  3. For each step S_i, we need to obtain an appropriate quantity Q_i that depends on the value vars at the S_i step. The quantity Q_i should not change from step to step. This makes it a genuine invariant of the process.

4. Using the initial state of the system we determine the value of the invariant.

5. By the invariant property, the value we found above should also coincide with the value at the final state.

6. Finally, knowing the value of the invariant at the final state should allow us to obtain significant info about the possible final states of the process.

- The most challenging part in applying this principle is finding the correct invariant. There is no general rule. It depends on the specific problem under consideration.

- Note: Some problems will require a generalized version of an invariant. I.e. We will need to create quantities Q for each step s.t. while the value changes from step to step, its parity/properties don't.

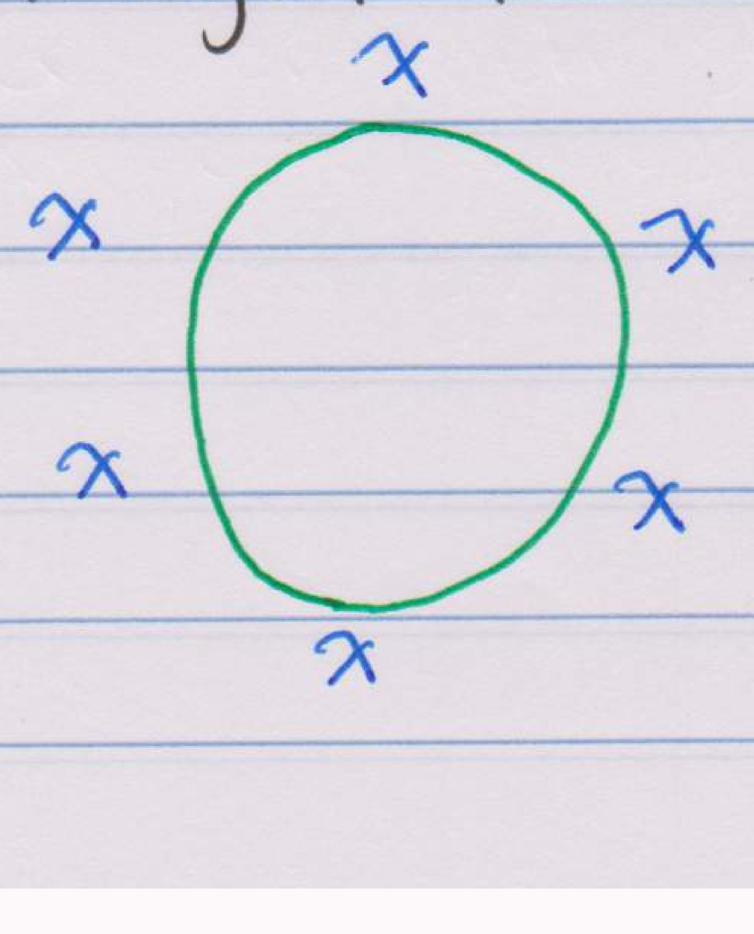
b) Examples

1) The nums 1, 2, 3, ..., 200 are written on the board. At each step:
   i) We select 2 nums that are written on the board, say x and y.
   ii) We delete x and y from the board.
   iii) We write $x+y$ on the board.

   What are the possible final nums on the board?

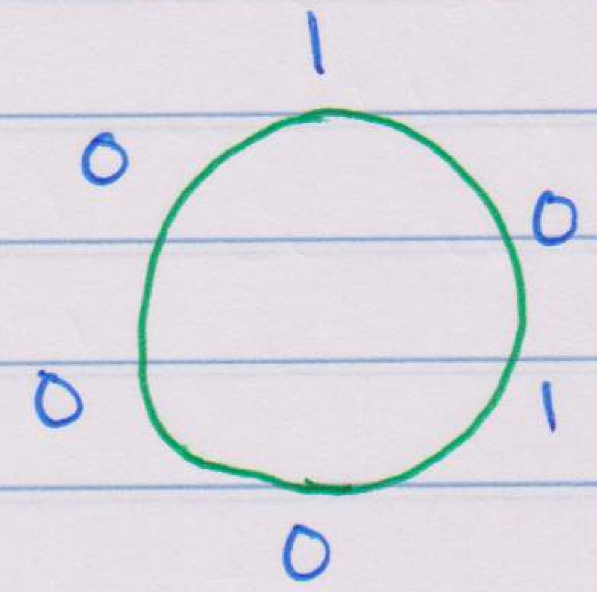
   Soln:
   Let Q be the sum of all the nums written on the board at each step. Clearly Q is an invariant, that is it does not change from step to step. Since Q is initially equal to 20100, the final num must be equal to 20100.

2) We write the nums 1, 0, 1, 0, 0, 0 on a circle shown below. At each step, we select 2 consecutive nums on the circle and add 1 to both of the selected nums. Is it possible to get the below state



I.e. Is it possible to get to a state where all 6 nums are equal?

Initial Circle



Soln:

Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the nums on the circle. Let $Q = x_1 - x_2 + x_3 - x_4 + x_5 - x_6$. This quantity is an invariant. To prove this, take any 2 consecutive nums and add 1 to them and replace the old vars with the new ones and simplify.

E.g. Suppose I choose x_3 and x_4

$$\begin{aligned} Q &= x_1 - x_2 + (x_3+1) - (x_4+1) + x_5 - x_6 \\ &= x_1 - x_2 + x_3 + 1 - x_4 - 1 + x_5 - x_6 \\ &= x_1 - x_2 + x_3 - x_4 + x_5 - x_6 \end{aligned}$$

Since $Q = 2$ initially, we can never reach a state where $Q = 0$ (I.e. when all the nums are equal.)

3) Can we put the nums $1, 2, 3, \ldots, 2019$ in a row to form a num that is a perfect square?

Soln:
Note that all the possible nums that we can form have the same digits and the same sum of the digits. This sum is 28140. 28140 is divisible by 3 but not by 9. However, any perfect square that is a multiple of 3 is also a multiple of 9. Therefore, we can't form a perfect square.

4) The nums $1, 2, \ldots, 10$ are written on a board. At each step,
a) We select 2 nums that are written on a board, say x and y.
b) We delete x, y from the board.
c) We write $x+y+xy$ on the board.
What are the possible final nums on the board?

Soln:
Note that $x+y+xy = (x+1)(y+1)-1$. This means that if we add 1 to all the nums written on the board and then take the product of the resulting nums, then this product is an invariant. Initially, this product is $11!$ So the end result must be $11!-1$.

5) Suppose there are the nums 1, 2, ..., 1000 written on a board. At each step, we select any 2 nums and replace them with their difference. The nums are randomly selected. Is it possible to have 243 as the only remaining num in the end?

Soln:

Let S_0 be the sum of the nums.

$S_0 = 500500$

After step 1, the sum is

$$S_1 = S_0 - a - b + (a - b)$$
$$= S_0 - 2b$$
$$= 500500 - 2b$$

Since both 500500 and 2b are even, S_1 is even. If S_1 is even, so will S_2, S_3, ..., S_n. Therefore, the final num can't be 243.